# A CRITIQUE OF THE ESSENTIAL PROPERTIES OF NUMBER IN THE REALIST THEORY OF MATHEMATICS

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***Abstract***
*This paper is written to evaluate the essential properties of numbers in the realist foundations of mathematics. The purpose of the study is to synchronize the achievements of schools of mathematics, especially realism (because of its non-vacuous status as well as its acceptance of the validity of first-order logic), with the fruitfulness of Peano Arithmetic (PA). The method adopted for the study is that of content analysis. It has unfortunately been found in the course of the study that realism is limited by its assumption of the validity of referential semantics. Hence, in order to avoid the Platonic consequences of its semantics, realism advocates an idea of the genetic essence of numbers as cardinals, thereby making the ordinal properties of numbers, which are the foundation of the fruitfulness of PA, an accidental product of the arrangement of cardinals. As a result, and very painfully, realism was unable to synchronize with PA*.
Keywords: Peano Arithmetic, Realism, Logicism, Empiricism, Semantics, and Number.

## Introduction

The use of the notion of number has assumed a normal dimension in everyday life. Number and enumeration have become so ingrained in human activities that their usage hardly gets noticed. Hence, what the concept means has been taken for granted by mathematicians and non-initiates alike. Great and fruitful mathematical systems have been built on this concept without a slight consideration of what its referent is. It was not until the 20th century that philosophers started asking concrete questions concerning the meaning of the concept. Several responses to this question emerged but were adjudged unsuccessful. It was Giuseppe Peano who developed an axiom system that gave scholars insight into what could be the real meaning of the notion. The system was built on three primitive, undefined notions, namely, zero, number, and successor.

The entities generated by the systems as numbers are serial entities with ordinal number properties. Hence, following Peano Arithmetic, numbers are essentially ordinals and become cardinal by application. The attraction for Peano Arithmetic lies in its mathematical fruitfulness, such that there is a branch of mathematical logic today called *PA,* or Peano Arithmetic. With such results, the paper assumes that *PA* could be taken as a standard for the evaluation of the performance of other proposals on the meaning of number in the foundations of mathematics. The proposal the paper has chosen to consider is the realist notion of number. Hence, it is the thesis of the essay that, following the achievement of Giuseppe Peano's (*PA*) axiom system, for the generation of numbers and its attendant mathematical fruitfulness, the notion of number is essentially ordinal and only cardinal in application. The essay is, therefore, purposed to test the realist theories of number to ascertain whether their definitions of number satisfy the ordinal essence of number as in Peano Arithmetic (*PA*).

## The Concept of Realism

The word "realism" is derived from the Latin root "realis," meaning "real." Further traces of the etymology of realism could be found in the interpretation of "res." "Res" is a Latin word often interpreted in philosophical literature to mean "substance." But the original meaning of the word "res" is "thing." Hence, it has the philosophical sense of meaning "a concrete thing" and "the real."

Realism originally appeared as a word to classify the mediaeval debate on the existence of universal objects. Consequently, the term "realism" was used within the context of the debate to contrast the terms conceptualism and nominalism, which represented different opinions in the debate on the problem of universals. The problem of universals involves the determination of whether there is a one-to-one correspondence between common terms in the intellect and things in the physical world. To this problem, there are three traditional divergent responses, namely: realism, nominalism, and conceptualism. Within the context of this debate, realism has been understood in two shades, namely, extreme realism and moderate realism.

Extreme realism, which is most often associated with Plato and the Platonists, is the argument that there is a parallel ontological relation between universal terms and actual objects in the universe. Consequently, the structure of physical objects represents the structure of intellectual objects. The argument of moderate realism, which is sometimes associated with Aristotle, is that universal concepts are the result of abstraction from physical objects.

## Theory of Platonist Mathematical Realism

Williard Quine has in recent times argued that the traditional mediaeval controversy concerning the problem of universals has resurfaced in discussions in the 20th century philosophy of mathematics (Quine, 1971, p. 13). According to Quine, realism is represented in modern-day logicism, while conceptualism is in intuitionism and nominalism in formalism, respectively (Quine, 1971, p. 14). Quine argues that, taking after the realist Platonist tradition, the mind can discover an abstract, independent realm of truth values. Logicism, championed by "Frege, Russell, Whitehead, Church, and Carnap, condones the use of bound variables to refer to abstract entities, known and unknown. . . "(1971, p. 14). Thus, they gave birth to a long tradition known as "realism" in the foundations of mathematics.

"Platonism is a word that refers to a family of doctrines" (Oliveri, 2007, p. 97) that believed that mathematical entities are not of the external world or mentalities but are abstract objects that exist in an abstract dimension. Platonism appeared to be based on the plausibility of the argument that "…if mathematics is a science of "relations of ideas" (Hume) or "relations between thoughts" (Frege), and not of matters of fact, then there is no reason to believe that abstract entities such as ideas or thoughts can be realised in concrete objects …" (Oliveri, 2007, p. 98). Abstract objects must remain abstract in their realm.

Platonism has traditionally held that numbers are abstract objects. Consequently, Platonists are of the opinion that mathematics, as number theory, is the study of the properties of numbers or present-day sets as abstract objects. The science of numbers is analogous to the science of physical objects, which studies the concrete perspectives of objects. The Platonist can therefore refer to the natural number independently and in relation to other numbers. In both cases, the properties of the number, say 3, will remain intact. Each number's unique properties as an abstract object determine the number's unique relationships to all other numbers. These properties and relations are the preoccupation of number theory as a science. Platonism faces the problem of the unity of its referents. For example, Paul Benaceraf demonstrated in set theory that Zermelo and von Neumann's expression of Peano's five axioms would face the problem of determining which unique set each of them refers to (Oliveri, 2007, p. 279). No one has access to the unique set, which is the abstract object modelled in all circumstances.

Frege is a notable contemporary Platonist. He argues that arithmetic is concerned with objects given in our "reason and transparent to it" (Oliveri, 2007, p. 100). Frege believed that we can define the identity of numbers within propositions. Frege rejected any condition of expression within which numbers are treated as predicates and not as objects.

By making numbers extensions of concepts, Frege gave them a life of their own and therefore conferred on them the status of objects. Numbers, according to Frege, occupy the third realm of ontology: "...thoughts are neither things in the external world nor ideas." "A third realm must be recognized" (Frege, 2007, pp. 17–18). Frege believes that objects in this realm have something in common with ideas (i.e., the fact that they cannot be perceived by the senses and are not the content of any person's consciousness). Consequently, they are true independently of whether someone acknowledges them or not, as thoughts need no owner (Frege, 2007, p. 17–18). Thoughts are analogous to planets; whether they are discovered or not, they are in interaction with other planets (Frege, 2007, p. 18). Just like the planets, we discover thoughts, but we do not create them.

Another prominent Platonist realist, Kurt Gödel, has argued that "... mathematical propositions... express properties of concepts" (Gödel, 2016b, p. 360). The argument leads to some firm realist ontological commitments, such as the fact that the properties of those concepts are as objective and independent of our choice as the physical properties of matter (Gödel, 2016b, p. 360). Gödel opposes every attempt to assume that the objects considered in mathematics, such as concepts, sets, and propositions, are man-made. This sort of thinking is, for him, wrong. He believes that: "… these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe" (Gödel, 2016a, p. 320). At the end of his Gibbs lecture, Gödel made an unreserved commitment to the Platonism of mathematical truths.

According to Gödel, the legitimacy of mathematical knowledge is only tenable as Platonism (Gödel, 2016a, pp. 322-323). Platonism was defined by him as "the belief that mathematics described a non-sensual reality that exists independently of both the acts and dispositions of the human mind and is only perceived in part by the human mind" (Gödel, 2016a, pp. 323–323). The entities discussed in mathematics belong to the second level of abstraction. First, by abstracting from individuals, we get their relations or concepts. Second, mathematics is concerned with the general properties of these concepts, and this belongs to the second level of abstraction. Gödel argues that we attain knowledge of this process through what he called "mathematical intuition."

## Theory of Aristotelian Mathematical Realism

Aristotle was not a philosopher of mathematics par excellence, but his idea of mathematical objects is realist, and this idea has influenced the philosophy of mathematics to a great extent and could be referred to as Aristotelianism. Aristotle understood mathematical entities to be the objects of perception, not independently but as abstractions from real things. Mathematical entities are, for Aristotle, objects of intelligence achieved through abstraction from perception. Oliveri (2007) presents his interpretation of Aristotle as follows: "...mathematical entities exist as intelligible possibilities, which are attributes of the object of perception, and whose knowledge can be attained by abstraction…" (p. 84). Mathematical entities are therefore attributes of things, not the things themselves. The Aristotelian approach to realism is perception and then abstraction. Its perceptual orientation excludes the notion of infinity from actual mathematics. Believing that infinity is at the heart of much of contemporary mathematics, it would be right to conclude that Aristotle's realist account is unsatisfactory.

Writers who have followed Aristotle's realism are J. S. Mill and Donald Gillies. Gillies (2000) argues that natural numbers and sets exist in the external world. The argument is predicated on the fact that Aristotle's acceptance of natural numbers as the properties of sets points to the fact that since sets are of the external world, natural numbers too are. Donald Gillies, falling back on the realist indispensability thesis of Quine and Putnam, argues that the infinite set is physically real if it is a component of a confirmed physical theory. On the same vein, he submits that Cantor's concept of the aleph (i.e., a number greater than $2^{N0}$) is non-physical and therefore metaphysical, since this concept lacks access and statements about it have no truth value. Hence, the truth value of an existential statement about mathematical entities is the truth value of any physical theory within which those entities are found. So, whether the mathematical entities named in a theory exist or not is dependent on whether the theory has been confirmed or not. If the theory is confirmed, then the entities exist in the physical world as properties of a set; otherwise, they do not exist in the exact relation stated by the theory. This leads to the realist arguments concerning the nature of mathematical statements, or what could be referred to as a "statement of number" (Gillies, 2000, p. 77).

## MATHEMATICAL EMPIRICISM AS REALISM

In the foundations of mathematics, realism of number is articulated by two major schools of philosophy of mathematics, namely empiricism and logicism. The discussion of their main theories and notions of number follows.

Foundational research is hooked up with genetic investigations of the existence and ontological status of theoretical entities. Such a genetic approach to knowledge is what empiricism represents. Empiricism is a theory of the legitimacy of knowledge and its object. As such, it is an epistemological movement, according to which:

1. Nothing around us can be known to be real unless its existence is revealed in or inferable from information we gain

directly in sense experience or in introspection of our subjective states, or latter recall, and

2. Genuine, intelligible differences in our claims about this world express this knowable difference in experience (Hunter 110).

The thesis statement of empiricism is that experience is the foundation of all knowledge. The movement found its major proponents in the seventeenth century in the hands of John Locke, George Berkeley, and David Hume. It has also won the admiration of several other scholars, like John Stuart Mill, Bertrand Russell, Rudolf Carnap, Williard Quine, etc. Among these latter scholars, Mill, Russell, and Quine contributed to the foundations of mathematics, but it was only Mill who went empiricist.

John Stuart Mill was a naturalist in his approach to the theory of knowledge. From a thoroughly naturalistic viewpoint, Mill understood the human person as part of the natural causal order studied by science. He thought that this had implications for knowledge: if minds are a part of nature, then no knowledge of the world can be a priori. Any assertion with real content must have an empirical basis. Mill thought that knowledge remained possible only on such a basis (Skorupski, 1993, p. 279).

This epistemology was extended to the foundations of mathematics, where he argued that mathematics is known inductively through experience (Resnik, 1980, p. 137). Thus, Mill contended that mathematical doctrines as well as all so-called deductive reasoning are in fact inductive. They are genetically rooted in the epistemology of physical objects (reality).

This epistemological conviction about the nature of mathematical theory caused Mill to reject both the conventionalist and the stipulationalist models of mathematical symbolism. On the contrary, he argued that mathematical theories refer to actual existence, and their definitions assert such existence. Thus, the postulates of arithmetic and geometry are empirically founded. Whereas the latter are approximately true of actual physical objects, the former are directly empirical. In his *System of Logic* (2006), Mill argues that "three is two and one" could be confirmed by observation and not by definition as with geometrical concepts (Book 1, Sec. 5, Para. 2). This does not mean that Mill attributes stipulationalism to geometry. It is Mill's conviction that, even though geometrical truths lack the absolute necessity of deductive theorems, they possess a relative empirical necessity.

Against philosophers and mathematicians like Leibniz, who thought that "arithmetic may be reduced via definition to mere identities," Mill taught that, in as much as true definition is not just verbal or abbreviations, arithmetic cannot in any way be reduced to mere identities (Resnik, 1980, p. 147).

Mill founded arithmetic in the empirical world, arguing that abstract algebra has some reference at each of its steps to real facts. Thus, all numbers, according to him, must be numbers of something. Ten is a number of bodies, sounds, pounds, etc. (Book 2, Sec. 4, Para. 2). As number of things, he understood it as a predicate term. Hence, we have ten bodies, ten sounds, ten pounds, etc. The numerals only give these predicates their perfect generality. Within this generalist understanding, ten denotes the particular entities to which it applies (i.e., all parcels or aggregates of physical objects). It is a property shared by all such aggregates. Mill's argument makes numbers appear to have discoverable and directly observable properties (Resnik, 1980, p. 149).

Arithmetical propositions are therefore not about abstract identities. Mill refers to them as "statements of the result of arithmetic operations; statements of one of the modes of formation of a given number" (Book 3, Sec. 24, Para 5). In this way, Mill rooted numbers in the structure of empirical aggregates or performed operations on such a structure. Hence, he defined numbers as ".... some property belonging to the amalgamation of things, which we call by the name; and that property is the characteristic manner in which the amalgamation is made up and may be separated into parts" (Book 3, Sec. 24, Para 5). The impression is completely graphic. A number is discovered not by counting but by mere observation. As a result, it has the appearance of being blue or white.

Mill's empiricist conception of number would no doubt reduce talk about ordinals to discourse about mere arithmetical contingencies. Hence, ordinality or seriality would be properly inessential to the existence of numbers. But the successor function makes it rather fundamental to Peano's Arithmetic. Even though Mill sometimes argues for the generation of numbers from series (i.e., by adding), such an idea is not basic to his thesis because he already allows for a non-serial possibility of numbers within the aggregate. Besides, he restricts the idea of serial analysis of numbers to education. He writes as follows: "We arrive at the conclusion (as all know who remember how they first learned it) by adding a single unit at a time; *5+1 = 6,* therefore, *5+1+1 = 6+1 = 7;* and again, *2* equals *1+1,* therefore, *5+2 = 5+1+1 = 7*" (Book 3, Sec. 24, Para. 5). So, the idea of a series is a function of creative operations with aggregates. Here, as in most foundations of mathematics, numbers lose their essence as ordinals and exist essentially as cardinals.

**Logicism as Realism**

The origin of the logicist realist tradition could be traced to G. Cantor. Cantor's reputation as a brilliant mathematician is rooted in his introduction of real numbers and the creation of set theory to handle problems associated with them. His main thesis on the theory of numbers is based on his theory of sets. One of the major relations in Cantor's set theory is the relation of equivalence. It was on the basis of this relation that he defined the cardinal number as follows: "Two sets *S* and *T* are said to be equivalent if there exists a one-to-one correspondence between them, i.e., if there is some relation such that each element of *S* is correlated by the relation with one and only one element of *T* correlated with it by the relation" (Kneale and Kneale, 1962, p. 439). Thus, Cantor defined the cardinal number or the power of a set as that which it has in common with all equivalent sets but with no others (Kneale and Kneale, 1962, p. 439). At the general level, the cardinal numbers are perceived as independent and are said to form the positive integers.

The above interpretation of integers as cardinal numbers of classes made the ordinal numbers dependent on the cardinals. Cantor therefore argues that all classes are well ordered. The well-ordering of a class is the foundation of a series or progression. Hence, he argues that ".... we can make each cardinal, which belongs to a well-ordered series, correspond to one and only one cardinal. Cantor assumes as an axiom that every class is the field of some well-ordered series and deduces that all cardinals can be correlated with ordinals.... (Russell, 1992, p. 322).

Cantor came so close to articulating the assumptions of this essay: that the distinction between cardinal and ordinal is ill-founded; that cardinals are ordinals understood independently of their genesis; that genetically, all numbers are ordinals and only cardinals when viewed as magnitudes. Surprisingly, Cantor never made such assertions. He instead asserted a distinction between cardinals and ordinals, thereby transforming ordinals into functions of cardinal arrangement. Consequently, Cantor opined that the fundamental property of number is cardinal. Unfortunately, unless a function concerning magnitude or any other property is defined on them, cardinal numbers lack the natural serial characteristics to establish succession in progression.

Another very prominent logicist is Gottlob Frege. The achievements of Frege in the foundations of mathematics marked the next great achievement after George Boole's algebra of logic. Whereas the latter sought to use mathematical apparatus in the analysis of logic, the former set out to prove that arithmetic was identical with logic. This implied that Frege would show how the ideas in arithmetic are only definable in terms of logic (Kneale and Kneale, 1962, p. 435).

Frege however, recognized that the old logic or any of its further independent developments was inadequate to bear the burden of his programme. As a result, he combined the accomplishments of Boole and the old logic with his newly constructed logic to establish a calculus of general reasoning. The kind anticipated by Leibniz, which, according to him, would free the human mind from the burden of natural language. He published the new theory in his *Concept Script* (Kneale and Kneale, 1962, p. 435). The development of notation in the *Concept Script* had two intentions: simplicity and demonstration, such that the laws that govern inferences in arithmetic are shown to be derived laws of logic (Macbeth, 2005, p. 17). "The goal is a system, a complete and adequate axiomatization of arithmetic; everything on which arithmetic proof depends is to be stated in advance, either as an axiom or a definition. . . ."(Macbeth, 2005, p. 17).

In this way, Frege claimed to have reduced the laws of arithmetic to those of logic, calling statements of both systems generalised conditions. Thus, numbers, laws of numbers, and operations on numbers would have logical definitions in his system. Frege began his logical analysis of mathematics by recognising the legitimacy of Leibniz's, Mill's, and others' arguments that the natural numbers greater than one are defined by reference to their predecessors (Kneale and Kneale, 1962, p. 455). As a result, "2 = 1 + 1," "3 = 2 + 1," and so on. But he remarked that such a position is incomplete so long as the number 1 and the notion of increasing by 1 are themselves undefined (Kneale and Kneale, 1962, p. 454). He also drew attention to the need for general propositions in order to develop arithmetic. Nevertheless, Frege criticised all views of numbers that reduced them to abstractions from physical objects. He therefore argued that "number is not abstracted from things in the way that colour, weight, and hardness are, nor is it a property of things in the sense that those others are" (Frege, 2007, p. 45).

Again, "number is not anything physical, nor is it anything subjective (an idea). Number does not result from the annexing of thing to thing . . . . " (Frege, 2007, p. 45). According to him, numbers can only be assigned to concepts as their properties and not to objects. He writes the following: “This is perhaps clearest with the number *0*. If I say, "Venus has 0 moons," there simply isn't any moon or agglomeration of moons for anything to be asserted of, but what happens is that the concept "moon of Venus" is given the property of including nothing under it” (Frege, 2007, p. 46). But Frege does not consider this definition satisfactory. According to him, a number as such is not a concept, even though in ordinary speech it is used as an adjective. It is not a property that any particular thing could be said to possess. "A number is not a concept but an object" (Kneale and Kneale, 1962, p. 457).

Frege does not also accept the view that a number is the common property of sets of the same size. Cantor championed this view. According to Kneale and Kneale (1962), the view has the benefit of simplicity (p. 458). Nevertheless, Frege argues that:

> *In the proposition "the number 0 belongs to the concept F", 0 is only an element in the predicate (taking the concept F to be the real subject). For this reason, I have avoided calling a number such as 0 or 1 or 2 a property of a concept. Precisely because it forms only an element in what is asserted . . . . "(2007, p. 57).*

Consequently, Frege believes that the individual number shows itself for what it is: a self-subsistent object (2007, p. 57). This object-oriented definition of "number" contradicts Frege's original argument, which identified the number as an adjective in natural language. He appears to accord more importance to the abstract existence of numbers in arithmetic than to their existence in daily usage.

Nevertheless, Frege’s definition of numbers is tied to his use of concepts. The definition uses the idea of the same number between two or more concepts to investigate the meaning of number. Consequently, Frege states that "the number that belongs to the concept *F* is the same as the number that belongs to the concept *G*" (Kneale and Kneale, 1962, p. 457). Frege wrote this way to avoid such a single statement as the number, which belongs to the concept *"F."* But if one were to examine the second case critically, it would be discovered that the only difference between them is their comparison in the previous instance. However, Frege argues that it was fashionable to write like that in his time (Frege, 2007, p. 63). Writers such as David Hume, George Cantor, and Bertrand Russell wrote in that manner. What such an approach purports is the idea of the independent existence of numbers.

To fulfil the objective, Frege built his definition on the basis of one-to-one correspondence between concepts, with number as the resultant object of the relationship. He argues that the objects falling under the two concepts *F* and *G* may be said to be correlated with each other by the relation $\phi$ if (i) every object falling under the concept *F* stands in the relations $\phi$ to an object falling under the concept *G,* and (ii) for every object falling under the concept *G*, there is an object falling under the concept *F* that stands next to it in the relation $\phi$. (iii) that for any *x*, *y,* and *z*, if *x* stands in relation $\phi$ to *y* and also to *z*, then *y* and *z* are the same, and (iv) that for *x*, *y,* and *z*, if *x* and *y* both stand in relation $\phi$ to *z*, then *x* and *y* are the same (Kneale and Kneale, 1962, p. 460).

On the basis of the presentation, Frege builds the following three definitions:

1. 'The concept *F* is like-numbered with the concept *G*' is to mean the same as 'there exists a relation *ɸ* which correlates the objects falling under the concept *F* on one-to-one with the objects falling under the concept *G*.'
2. The number which belongs to the concept *F* is the intension of the concept 'like-numbered with the concept *F*.'
3. '*n* is a number' is to mean the same as 'there exists a concept such that *n* is the number which belongs to it' (Frege, 2007, p. 72)

Frege believed that his definition has twofold achievements: it defines numbers and overcomes the use of the circular concept of a one-to-one relation. Hence, he argued that the definition was logical. Kneale and Kneale (1962) present this argument in the following way: "... concepts are to be called like-numbered if their extensions are equivalent according to the technical terminology of the theory of sets" (p. 460). The argument of Kneale and Kneale (1962) implies that Frege's definition of number is identical with Cantor's notion of cardinals.

The idea of numbers as the "number concept" is central to Frege's analysis. Even when he transferred the whole analysis into a logical presentation of series, which is necessary for the justification of arithmetic, he still retained, as essential, the notion of "concept number," which is identified with Cantor's idea of cardinal. Whatever the fate of Cantor's cardinals is in this essay, it therefore applies to Frege's numbers. His series consists of the following:

*0* is a number belonging to the concept "not identical with itself."

*1* is a number belonging to the concept "identical with *0*."

*2* is a number belonging to the concept 'identical with *0* or *1*'

*3* is a number belonging to the concept 'identical with *0* or *1* or *2*'. . . . (Kneale and Kneale, 1962, p. 466)

One thing lacking in the definition of "series" is the necessity of the concept names used. The presentation is therefore a contingent and not a necessary model of number theory. The only difference between it and intuitionist interpretations is that zero is absent in the latter's model, and numbers are understood as temporal instances. From the traditional understanding of numbers as cardinals, the result of Frege's analysis would be as follows: "Zero is a null set; one is a set with an element; two is a set with two elements, etc." The basic limitation of the above series is its inability to establish a justification for the necessity of its concept types and the ground on which a series is necessarily a derivation of the type here presented. The only warrant could be the contextualization of numbers within the concept of "a controlled moment of cognition or realisation from zero upward." But why would Frege basically choose that concept and not another? Besides, this concept does not possess any privilege over any other concept.

For instance, zero could mean the class of an actual round square in the physical world, and one could mean the class of "the German President," who ruled Germany during the Second World War. Just in that sense of looking for specific intentions, we can fill in all the natural numbers and then begin to put them in a series in the order of increasing magnitude.

Consequently, the property of

identity would simply turn out to be one such instance of recognising numbers. Identity does not create a number. It matches two concepts or properties possessing something in common. The use of the concept of identity does not reduce a system to logic. After all, it has been used in physics, biology, and even in everyday parlance. Raatikainen (2003, p. 162) has drawn this implication for logicism from his analysis of Sternfeld's work on "The Logistic Thesis" (1976, p. 147) and Rodriguez-Consuegra's "Russell, Gödel, and Logicism" (1993, p. 175).

Besides, what does it mean to say that the number zero is the number that belongs to the concept *F?* What kind of circular definition is that supposed to be? Hence, before the formation of series was to become an issue in the essay, the definition of numbers itself was problematic. Again, how could the idea of "identical with one or zero" be applied in a real-life situation? Frege's system makes numbers so complex and unrealistic. How do we explain to someone that a class with a zero always represents a one? What a contradiction! After all, there are millions of contradictions in the world, such that if they were referred to as numbers, the next number after zero would not be one but trillions. Again, which contradiction would generate *one* such that no other ever would? It is important to note that Frege presupposed the validity of ordinary arithmetic before proceeding to prove it. The system leaves so much to be desired, especially in the areas of series and number definition. Consequently, it fails the test set forth in the essay.

Frege's analysis would have been so exciting if numbers were taken as states of knowledge, which is what they are for him. But he did not recognise this feat. For instance, if zero had been a contradiction (i.e., the class of *"a"* contradiction), the recognition of that "specific instance" of contradiction (because there are infinitely many possible contradictions) would have been *1* and so on. Even though that description would have been more meaningful, it would still be lacking in necessity as a model for number theory. It would simply be an application of number-theoretic truths and not their essence. Nevertheless, the seeming contradiction arising from the derivation of *1* from zero would have been avoided. The impetus for Frege to state that zero is a contradiction and 1 is zero, as if there is only one contradiction, is a type of Platonism that views numbers as strange entities.

An elaborate logicist attempt to essentially model the concept of number was made by Bertrand Russell in his *Principles of Mathematics* (1992). He began with the conviction that the programme of logical analysis in mathematics had reached an advanced stage in the works of Peano. He traces the evolution of the programme from the reduction of all traditional pure mathematics to the theory of natural numbers and to the stage characterised by the reduction of this theory to a handful of principles and undefined terms by Peano (Russell, 1998, p. 5). The consideration that made it possible to advance beyond Peano and further reduce the mathematical theory to simpler and more general notions was attempted by Gottlob Frege (Russell, 1998, p. 7). This condition included indications that mathematics could be reduced to logic or a system of simple general principles rather than the ones proposed by Peano previously. Such a system would itself become a model for Peano's mathematics. Russell's number model was an attempt to provide a logical interpretation of Peano's mathematics in particular and all of pure mathematics in general; after all, Peano's mathematics is just a document or a reduced model of all mathematics.

Three fundamental logical concepts were prominently featured in Russell's program: the concept of cardinal number, the concept of ordinal number, and the concept of series. The notion of cardinality evolved from Cantor's development of set theory for the analysis of real numbers. According to Cantor, the cardinal number of a set represents the numerical capacity of its elements. Thus, the set of "boys living in Room *10*" would have the cardinal number "three" *(3)* if the boys are John, James, and Andrew. So, a cardinal number is the number of elements in a set. Numbers as such are generalization on cardinal. They are classes.

Thus, Russell argues that numbers are classes of classes. The classes of which they are classes are those with the same numerical strength. In his discussion of how numbers model Peano's system, he writes as follows:

> *According to my theory, the class of classes satisfying his axioms is the same as the class of classes, which is $\alpha_0$. It is most simply defined as: c is the class of classes u, each of which is the domain of some one-one relation R (the relation of a term to its successor) which is such that there is at least one term which succeeds no term, every term which succeeds has a successor and u is contained in any class, which contains a term of u having no predecessors, and also contains the successors of every term of u which belongs to s (Russell, 1992, p. 127).*

Russell's *u* and *s* are finite, whatever numbers they are. So, what Russell means here is that the class of finite numbers satisfies Peano's mathematical axioms. This conception has implications for the primacy of cardinal numbers in mathematical thinking because the finite numbers in question are all cardinal.

To be sure, $\alpha_0$ is a class of cardinal numbers. As a result, the ordinal property of numbers is a construct imposed on them. Russell does not also deny the fact that cardinal numbers are capable of ordinal behaviour. But he is insistent on the conviction that the ordinal property of numbers is a function of the serial arrangement of cardinal numbers. In that sense, numbers are not necessary for series, and vice versa. He argues for the independence of each from the other. According to him, "natural numbers are a particular case of . . . series, and . . . the whole of (mathematics) could be developed out of any one of such series, without any appeal to number . . . ."(Russell, 1992, p. 239). Thus, Russell argues for a definition of progression without appealing to numbers.

The larger implication of this analysis is that numbers are not necessary for mathematics. Any entity fulfilling the properties of numbers could act as a capable substitute. Russell believes that classes possess such capabilities. On the basis of this, he believed that mathematics could be reduced to logic or to logic and set theory.

The contention that reduces numbers to classes also deprives them of their ordinal essence. Thus, Russell argues, the cardinal properties of numbers are prior to the ordinal properties. The ordinal is a serial arrangement of cardinals. Even though he admits that the ordinals can be referred to without referring to the cardinal, he contends that the numerical property of the ordinal is a function of their being conceived as terms of the cardinals. In whichever case, the ordinals are numerically inessential numbers.

So Russell arrived at Peano's mathematical achievement by first interpreting numbers in terms of classes. But if numbers are classes and the classes representing those numbers represent the numerical strength of first-level classes, how is the number of such classes to be determined or measured without the use of counting or the ordinal property? This question also faced Russell, but he dismissed it as though it were not a problem. According to him, the determination of a number as an equivalent class of classes is done by correlating such classes using the principles of one-to-one correspondence. It is difficult to explain how such a correlation would be carried out and the number determined without counting. Besides, the use of the concept of "one-to-one" in the definition makes it circular. One, for instance, is a number, and it is circular to try to define a number by itself.

Russell recognized this circularity and therefore proceeded to argue that even though the definition appears circular, it is not, in fact, circular. According to him, some definitions are that way and are also unproblematic because they are simply clear. However, it is unclear what he means when he says "it is circular but not in a certain sense." Such definitions as "the number of a class is the number of the elements of the class" are expressions of improper definition. If modelling means providing meaning for a system, then a definition is an inadequate model of the system it purports to model if it is an improper definition of the system. Such is the fate of logicism in general and Russell's logicism in particular. Russell had to face the problem because of his refusal to recognize the essentiality of series (counting) in mathematical modeling.

**Summary and Conclusion**

The central element of Peano Arithmetic's (*PA*) achievement for epistemology, as shown in the essay, is the location of the genetic foundations of number within the framework of a series. It shows that, genetically and essentially, numbers are ordinals and then cardinals in application. This achievement forms the legitimating myth for the evaluation of realism in the paper.

Realism is shown in the essay to be founded on the legitimacy of referential semantics. This semantics is referred to as referential realism in this context. Referential realism is an advocate of the extra-mental existence of concepts' referents, like numbers. So, for realism to satisfy the *PA*-type foundations of number, it must show some extra-mental existence of series or progression as its genetic basis. Such an assumption would lead to bizarre Platonism, a problem the realists want to avoid. To completely avoid the unfortunate Platonic consequences of their thesis, realism argues in the paper for the primacy of numbers as genetically cardinal, thereby making the ordinal property of numbers an accident of the arrangement of cardinals.

In conclusion, therefore, the study found out that realism fails to defend the ordinal genesis of numbers, which genesis is the epistemic ground for the mathematical fruitfulness of *PA*. But most importantly, it was discovered that the failure of realism to establish the ordinal essence of numbers arises from the limitation of its referential semantics, which obviates subjective inputs from foundational considerations.

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